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Solutions of the Maxwell and Yang-Mills Equations Associated with Hopf Fibrings

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Abstract

It is shown that the magnetic pole of lowest strength and the pseudoparticle solution of the Yang-Mills equations correspond to natural connections defined on the principal bundles $U(2)/U(1) = S_3 \to S_2$ and $Sp(2)/Sp(1) = S_7 \to S_4$, respectively. This observation leads to a general method of constructing new, topologically nontrivial solutions of the Maxwell and Yang-Mills equations. Among them is an "electromagnetic instanton" defined over the two-dimensional complex projective space endowed with the Fubini-Study metric.

Recent theoretical work on the properties of magnetic poles (Nambu, 1974; Parker, 1975; Goldhaber, 1976; Wu and Yang, 1976; many references are given by Goldhaber and Smith, 1975) and on the Yang-Mills instanton (Belavin et al., 1975; Hooft, 1976a, b; Jackiw and Rebbi, 1976a; Callan et al., 1976) encouraged me to consider the geometrical models that can be associated with the corresponding classical gauge fields. It is known that electromagnetism and the Yang-Mills theory admit an interpretation in terms of connections and curvatures on principal bundles with the structure groups $U(1)$ and $SU(2)$, respectively (Yang and Mills, 1954; Lubkin, 1963; Trautman, 1970). Clearly, the $U(1)$ bundle carrying a connection corresponding to a magnetic pole is nontrivial (Wu and Yang, 1975; Ezawa and Tze, 1976). Consider a magnetic pole at rest relative to an inertial frame in Minkowski space-time R^4; the manifold R^4 with the worldline of the pole removed is diffeomorphic to $R^2 \times S_2$. One is thus led to consider circle bundles over S_2; they are all known. The "simplest," nontrivial among them was described by Hopf (1931) in the same year Dirac (1931) published his paper on magnetic poles.

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Let z_0, z_1 be two complex numbers, then $\bar{z}_0 z_0 + \bar{z}_1 z_1 = 1$ defines a three-dimensional sphere S_3. The group $U(1)$ acts on S_3 by $(z_0, z_1)u = (z_0 u, z_1 u)$, where $u \in U(1)$, i.e., $\bar{u}u = 1$. The orbits (fibers) of $U(1)$ in S_3 are circles and the quotient of S_3 by this action is S_2. The projection $S_3 \to S_2$ is given by a composition of $(z_0, z_1) \mapsto z_1/z_0$ with the stereographic map $C \to S_2$. This Hopf fiber bundle admits a natural connection, which may be conveniently expressed in terms of the Euler angles: Set

$$z_0 = [\exp \tfrac{1}{2}i\,(\chi + \phi)]\,\cos \tfrac{1}{2}\theta, \qquad z_1 = [\exp \tfrac{1}{2}i\,(\chi - \phi)]\,\sin \tfrac{1}{2}\theta$$

and compute the Riemannian line element of S_3,

$$4(d\bar{z}_0 dz_0 + d\bar{z}_1 dz_1) = d\theta^2 + \sin^2\theta\, d\phi^2 + (d\chi + \cos\theta\, d\phi)^2$$

The form $\alpha = \frac{1}{2}(d\chi + \cos\theta\, d\phi)$ defines a connection on S_3 considered as a circle bundle over S_2. Its curvature $F = \frac{1}{2}\sin\theta\, d\phi \wedge d\theta$, extended to Minkowski space-time is the electromagnetic field of a magnetic pole of strength $g = \frac{1}{2}$. (The units are such that the charge of the electron is equal to the fine-structure constant). The form α is smooth and invariant under the transitive action of $U(2)$ on S_3. The singularities of the potentials of the magnetic pole are due to the nontrivial character of the bundle $S_3 \to S_2$. The map s, sending S_2, with the north pole ($\theta = 0$) removed, into S_3, and defined by $s(\theta, \phi) = (z_0 = e^{i\phi} \cos \frac{1}{2}\theta, z_1 = \sin \frac{1}{2}\theta)$ is smooth, but it cannot be extended throughout S_2. Therefore, s is only a local section and the potential A in the gauge s, $A = s^*(\alpha) = \frac{1}{2}(1 + \cos\theta)d\phi$, is singular at $\theta = 0$ because its essential component with respect to an orthonormal frame is $A_\phi = (1 + \cos\theta)/2r\sin\theta$.

The above construction may be generalized by considering multidimensional spaces and allowing the coordinates $z_\alpha \in K$ to be either complex ($K = C$) or quaternionic (Finkelstein et al., 1973) ($K = H$). The equation

$$\bar{z}_0 z_0 + \bar{z}_1 z_1 + \cdots + \bar{z}_n z_n = 1 \tag{1}$$

defines an S_{2n+1} or an S_{4n+3}, depending on whether $K = C$ or H. The group $G(n+1)$ of linear, K-valued transformations acting on the z's on the left and preserving the quadratic form (1) is $U(n+1)$ in the first, and $Sp(n+1)$ in the second case (Steenrod, 1951; Husemoller, 1966). The group $Sp(1)$ of unit quaternions is isomorphic to $SU(2)$. In either case, the group $G(1)$ acts freely on the sphere (1) by $(z_0, \ldots, z_n)u = (z_0 u, \ldots, z_n u)$, $u \in G(1)$. The quotient of (1) by this action is the projective space in n dimensions over K. There are thus two sequences of Hopf principal fiber bundles:

$S_{2n+1} \to CP_n$ with group $U(1)$

$S_{4n+3} \to HP_n$ with group $Sp(1) = SU(2)$

Assuming $z_0 \neq 0$ one can introduce a local trivialization of the sphere (1) by writing $z_0 = \rho u$ and $z_a = \zeta_a z_0$, where $\rho = |z_0| > 0$ and $a = 1, \cdots, n$. It follows from these definitions that $u \in G(1)$ and $\rho^{-2} = 1 + \Sigma_a \bar{\zeta}_a \zeta_a$. The

ζ's constitute a local coordinate system on the projective space. The Riemannian line element on the sphere is

$$dl^2 = \sum_{\alpha=0}^{n} d\bar{z}_\alpha dz_\alpha$$

and may be computed in terms of u and ζ_a:

$$dl^2 = ds^2 - \omega^2$$

where

$$\omega = u^{-1}du + \tfrac{1}{2}\rho^2 u^{-1} \sum_a [\bar{\zeta}_a d\zeta_a - (d\bar{\zeta}_a)\zeta_a]u$$

and ds^2 is the symmetric part of the positive definite Hermitean form

$$\sum_{a,b} d\bar{\zeta}_a h_{ab} d\zeta_b$$

with $\bar{h}_{ab} = h_{ba}$ given by

$$\Omega = d\omega + \omega \wedge \omega = u^{-1} \sum_{a,b} (d\bar{\zeta}_a \wedge h_{ab} d\zeta_b)u \qquad (2)$$

The forms $u^{-1}du$, ω and Ω have values in the Lie algebra of $G(1)$, i.e., in the pure imaginary subspace of K. Therefore, the quadratic form $-\omega^2$ is positive definite. Since both the latter form and dl^2 are invariant under the action of $G(1)$, so is ds^2 and it defines a Riemannian metric on the projective space. In the complex case, $\omega \wedge \omega = 0$, and, if one writes $\omega = i\alpha$, $\Omega = iF$, then both α and F are real, and F is the Hodge form (Weil, 1958; Chern, 1967; Morrow and Kodaira, 1971) of CP_n.

The fundamental result of this paper is that, for any n, Ω given by (2) is a solution of the source-free Maxwell ($K = C$) or Yang-Mills ($K = H$) equations, invariant under $SU(n + 1)$ or $Sp(n + 1)$, respectively. To prove this, we note that Ω satisfies the Bianchi identity,

$$D\Omega = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0$$

and is invariant under $G(n + 1)$ by construction. The $2n$ form $F \wedge \cdots \wedge F$ (n factors) is a volume element on CP_n, whereas the $4n$ form $\Omega \wedge \cdots \wedge \Omega$ ($2n$ factors) plays a similar role on HP_n. These volume elements define orientations which, together with ds^2, determine the duals of differential forms. The dual $*\Omega$ of Ω is proportional to $\Omega \wedge \cdots \wedge \Omega$, where the exterior product contains $n - 1$ factors for CP_n and $2n - 1$ factors for HP_n. Therefore, the Bianchi identity implies that the gauge field Ω is source-free:

$$D{*\Omega} = 0$$

For example, the Belavin-Polyakov-Schwartz-Tyupkin solution corresponds to $K = H$ and $n = 1$: There is then one quaternion coordinate ζ, $\rho^{-2} = 1 + \bar{\zeta}\zeta$, and

$$ds^2 = \rho^4 d\bar{\zeta} d\zeta$$

is the line-element of a four-dimensional sphere of radius $\frac{1}{2}$. The local section $u = 1$ leads to the potential $\frac{1}{2}\rho^2 [\bar{\zeta} d\zeta - (d\bar{\zeta})\zeta]$ and the field $\rho^4 \, d\bar{\zeta} \wedge d\zeta$. The action of $Sp(2)$ on S_7 projects to an action of $SO(5)$ on $HP_1 = S_4$ and the solution is invariant under the latter group (Jackiw and Rebbi, 1976b, Yang, 1977).

A new solution of Maxwell's equations is obtained for $K = C$ and $n = 2$. In local coordinates on CP_2 given by $\zeta_1 = e^{i\mu} \tan\theta \cos\phi$, $\zeta_2 = e^{i\nu} \tan\theta \sin\phi$, the electromagnetic field is

$$F = \sin 2\theta \, d\theta \wedge (\cos^2\phi \, d\mu + \sin^2\phi \, d\nu) - \sin^2\theta \sin 2\phi \, d\phi \wedge (d\mu - d\nu) \quad (3)$$

whereas the Fubini-Study metric assumes the form

$$ds^2 = d\theta^2 + \sin^2\theta \, [d\phi^2 + \cos^2\theta \, (\cos^2\phi \, d\mu + \sin^2\phi \, d\nu)^2 + \sin^2\phi \cos^2\phi \, (d\mu - d\nu)^2] \quad (4)$$

The field (3) is self-dual, $^*F = F$, and its energy-momentum tensor vanishes. Therefore, equations (3) and (4) define a solution of Einstein's equations with a cosmological term. Following a suggestion by Eguchi and Freund (1976), this solution, which is invariant under $SU(3)$, could be called the gravitational and *electromagnetic instanton.* The integral $\int F \wedge F$ associated with the second Chern class is equal to $4\pi^2$.

If X is an analytic submanifold of CP_n, then the embedding $k : X \to CP_n$ may be used to pull the Hodge form F from CP_n back to X and to define thus a new solution of Maxwell's equations on X. For example, for any positive integer n there is an embedding $k_n : S_2 = CP_1 \to CP_n$ given in terms of the homogeneous coordinates (z_α) by

$$k_n(z_0, z_1) = (z_0{}^n, \binom{n}{1}^{1/2} z_0^{n-1} z_1, \ldots, \binom{n}{m}^{1/2} z_0^{n-m} z_1^m, \ldots, z_1{}^n)$$

An electromagnetic field pulled by k_n from CP_n to S_2 corresponds to a magnetic pole of strength $g = n/2$. Moreover, k_n induces over S_2 a circle bundle isomorphic to the lens space $L(n, 1)$ (Greenberg, 1967).

An interesting possibility, now under investigation, is to generalize the method described in this paper to spaces with an indefinite metric, by replacing the groups $U(n)$ and $Sp(n)$ by $U(p, q)$ and $Sp(p, q)$, respectively.

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